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PB$^\infty$-SUBGROUPS OF BUTLER GROUPS*

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ABSTRACT

It is well known [BF] that in the constructible universe (V = L) the class of B_1-groups is closed under prebalanced subgroups. A similar attempt at the increasing countable unions of prebalanced subgroups has been done in [B]. Here we give an affirmative answer to a similar question concerning the so-called PB^∞-subgroups.

Introduction

All groups in this paper are abelian. If x is an element of a torsion-free group G then $|x|_G$, or simply $|x|$, is the characteristic, and $\mathbf{t}_G(x) = \mathbf{t}(x)$ is the type, of x in G. The letter G will usually denote a general torsion-free group, while the letter B will be used for Butler groups. For unexplained terminology and notation see [F]. As usual, (GCH) denotes the **generalized continuum hypothesis**, i.e. $2^\kappa = \kappa^+$ for each infinite cardinal κ. By a **smooth (increasing) union**

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$\bigcup_{\alpha<\lambda} G_\alpha$ **of a group** G, we mean the union of a collection of pure subgroups G_α indexed by an initial segment of ordinals with the property that $G_\beta \le G_\alpha$ when $\beta < \alpha$ and $G_\alpha = \bigcup_{\beta<\alpha} G_\beta$ whenever α is a limit ordinal. Especially, by a **countable union of a group** G is meant any union $G = \bigcup_{n<\omega} G_n$ of pure subgroups, where $G_n \le G_{n+1}$ for each $n < \omega$.

An exact sequence E: $0 \to H \to G \xrightarrow{\beta} K \to 0$ with K torsion-free is **balanced** if the induced map β_*: $\mathrm{Hom}(J, G) \to \mathrm{Hom}(J, K)$ is surjective for each rank one torsion-free group J. Equivalently, E is balanced if all rank one (completely decomposable) torsion-free groups are projective with respect to E. A torsion-free group B is said to be a B_1**-group** (**Butler group**) if $\mathrm{Bext}(B, T) = 0$ for all torsion groups T, where Bext is the subfunctor of Ext consisting of all balanced-exact extensions. It is known [BS] that this definition coincides with the familiar one if B has finite rank, i.e. a pure subgroup of a completely decomposable group, or, equivalently [Bu], a torsion-free homomorphic image of a completely decomposable group of finite rank.

Recall that a subgroup H of a group G with a torsion-free quotient G/H is called **prebalanced** if for each rank one (pure) subgroup J of G/H there is a pair (X, ϕ) consisting of a finite rank completely decomposable group X and a homomorphism ϕ: $X \to G$ such that $\beta\phi X = J$, β being the canonical projection $G \to G/H$. Equivalently (see [FMe]), a pure subgroup H of a torsion-free group G is prebalanced if and only if for each $g \in G$ there are a non-zero integer m and a finite subset $\{h_0, \ldots, h_n\} \subseteq H$ such that $\mathbf{t}(g + H) = \bigcup_{i=0}^{n} \mathbf{t}(mg + h_i)$. An exact sequence $0 \to H \xrightarrow{\alpha} G \xrightarrow{\beta} K \to 0$ is prebalanced if αH is a prebalanced subgroup of G. For the sake of brevity, we shall say that a subgroup H is G-pure if it is pure in G. Similarly we shall use the terms G-balanced, G-prebalanced, etc.

Another relevant concept in the study of infinite rank Butler groups is the **torsion extension property** (TEP). A pure subgroup H of a torsion-free group G is said to have TEP in G, or briefly, H is TEP(-subgroup) in G, if every homomorphism $H \to T$ with T torsion extends to a homomorphism $G \to$T.

It is well known [BF; Th.4.5] that, in the constructible universe, the prebalanced subgroups of B_1-groups are B_1-groups and the same (cf. [DHR; Cor.5.9]) holds for groups of cardinalities $\le \aleph_\omega$ under (CH). Moreover, in [B] a condition (PB_κ) ensuring the existence of "enough prebalanced subgroups" for groups up to the cardinality κ has been introduced and it was proved that under this condition countable unions of prebalanced subgroups of B_1-groups are B_1-groups for the groups of cardinalities up to κ. As a corollary one obtains the same result for groups of cardinality at most $\aleph_\omega$ under a weaker hypothesis (CH).

Since any smooth increasing union $\bigcup_{\alpha<\lambda} H_\alpha$ of prebalanced subgroups of a torsion-free group G is prebalanced whenever cof $\lambda \neq \omega$, this result gives a solution for each smooth increasing union of prebalanced subgroups.

Dugas, Hill and Rangaswamy [DHR] introduced the class of B^{∞}-subgroups of a torsion-free group G as a smooth union of the classes of B^{μ}-subgroups for $\mu < \omega_1$, where the class of B^0-subgroups consists of all balanced subgroups of G and the class of $B^{\mu+1}$-subgroups is formed by taking unions of increasing countable chains of B^{μ}-subgroups. Replacing in this definition the balanced subgroups by the prebalanced ones, we obtain the class of PB^{∞}-subgroups. Owing to the construction of this class it is natural to ask whether the members of this class in a B_1-group are again B_1-groups.

The purpose of this paper is to give an affirmative answer to this question in the constructible universe (V = L) for general groups, and, under (CH), for groups of cardinalities $\leq \aleph_\omega$. But if G is a B_1-group with $|G| \leq \aleph_1$, then no additional set theoretic hypothesis is needed and we show, under ZFC, that a PB^{∞}-subgroup (indeed, any preseparative subgroup) of G is again a B_1-group.

There are two substantial techniques used in the proof of the results mentioned above. The first one, used by Fuchs and Magidor in [FMa], is the exploitation of Jensen's box principle $\Box_\kappa$, while the second one, developed by the first author in [B], combines the construction of prespecial subsets with the construction of prebalanced subgroups given in [DHR].

1. Preliminaries

First, we shall collect some results which will be useful in the sequel.

1.1 Lemma: *If $0 \neq H \leq G$, then there is a balanced subgroup L of G with $H \leq L$ and $| L | \leq | H |^{\aleph_0}$.*

Proof: See [DHR; L.5.2]. ∎

1.2 Lemma: *Let λ be a limit ordinal and $H = \bigcup_{\alpha<\lambda} H_\alpha$ be a smooth increasing union such that $H_{\alpha+1}$ is prebalanced in G for all $\alpha < \lambda$. If cof $\lambda \neq \omega$ then H is prebalanced in G.*

Proof: See [B; L.2.4]. ∎

1.3 LEMMA (GCH): *Let G be a torsion-free group and $X \subseteq G$ be a subset of infinite cardinality not cofinal to ω. Then there is a (pre)balanced subgroup H of G such that $X \subseteq H$ and $|X| = |H|$.*

Proof: For $|X|$ regular it suffices to use Lemmas 1.1 and 1.2. So, let $|X| = \sum_{\alpha<\lambda} \kappa_\alpha$ be singular, where $\lambda = \mathrm{cof}\, |X| > \omega$ and $\kappa_0, \kappa_{\alpha+1}$ are regular for each $\alpha < \lambda$. Write X as a smooth union $X = \bigcup_{\alpha<\lambda} X_\alpha$, where $|X_\alpha| = \kappa_\alpha$ for each $\alpha < \lambda$. By Lemma 1.1, we take H_0 to be a prebalanced subgroup of G containing X_0 and having cardinality κ_0. Let $\alpha < \lambda$ be arbitrary and assume that for each $\beta < \alpha$ the subgroup H_β of cardinality κ_β containing X_β has been constructed in such a way that $H_\beta = \bigcup_{\gamma<\beta} H_\gamma$ for β limit and $H_{\beta+1}$ is prebalanced in G whenever $\beta + 1 < \alpha$. For α limit we simply set $H_\alpha = \bigcup_{\beta<\alpha} H_\beta$, while for $\alpha = \beta + 1$ we select a prebalanced subgroup H_α of G of cardinality κ_α containing $H_\beta \cup X_\alpha$ Setting $H = H_\lambda = \bigcup_{\alpha<\lambda} H_\alpha$ we see that $X \subseteq H$, $|X| = |H|$ and H is G-prebalanced by Lemma 1.2. ∎

1.4 LEMMA: *Let E: $0 \to A \to B \to C \to 0$ be an exact sequence of torsion-free groups.*

(i) *If E is* TEP *and B a B_1-group, then C is a B_1-group;*
(ii) *if E is prebalanced and C is a B_1-group, then E is* TEP.

Proof: See [FMe; Prop. 2.1]. ∎

1.5 PROPOSITION: *Let $B = \bigcup_{\alpha<\lambda} B_\alpha$, λ a limit ordinal, be a smooth union of pure and B_1-subgroups of a torsion-free group B. If, for each $\alpha < \lambda, B_\alpha$ is* TEP *in $B_{\alpha+1}$, then B is a B_1-group.*

Proof: See [BB; Prop. 2.2]. ∎

2. PB^∞-subgroups and Hill's compatibility

The following class of subgroups in the balanced case has been introduced and investigated in [DHR].

2.1 Definition: Let G be a torsion-free group. By induction on ordinals $\mu < \omega_1$, we define the PB^μ-subgroups of G. The PB^0-subgroups of G are precisely the prebalanced subgroups of G. If $\mu < \omega_1$ is a limit ordinal, then PB^μ-subgroups are all the subgroups of G that are PB^ν-subgroups for some $\nu < \mu$. The $PB^{\mu+1}$-subgroups are the unions of countable increasing chains of PB^μ-subgroups. A subgroup H of G is called a PB^∞-subgroup if it is a PB^μ-subgroup for some $\mu < \omega_1$. For this situation we shall also use the notation $H \in \mathcal{PB}^\mu(G)$ or $H \in \mathcal{PB}^\infty(G)$.

2.2 Definition: Let G be a torsion-free group. For $K \in \mathcal{PB}^\infty(G)$, define a (countable) collection $\mathcal{M}(K)$ of subgroups of G inductively as follows:

(1) if $K \in \mathcal{PB}^0(G)$ then we set $\mathcal{M}(K) = \{K\}$;

(2) if $\mu < \omega_1$ and $K \in \mathcal{PB}^{\mu+1}(G)$ say, $K = \bigcup_{n<\omega} K_n$ with $K_n \in \mathcal{PB}^\mu(G)$, then we set $\mathcal{M}(K) = \{K\} \cup \bigcup_{n<\omega} \mathcal{M}(K_n)$.

If $\mathcal{K}$ is any collection of PB^∞-subgroups of G then we set $\mathcal{M}(\mathcal{K}) = \bigcup_{K\in\mathcal{K}} \mathcal{M}(K)$.

Finally, if $H \leq G$ and $\mathcal{K}$ is any collection of PB^∞-subgroups of G, then $H + \mathcal{M}(\mathcal{K})$ means $\{H + K \mid K \in \mathcal{M}(\mathcal{K})\}$.

2.3 Definition: If K is a PB^∞-subgroup K of a torsion-free group G, then $\mathrm{d}(K)$ denotes the first (necessarily non-limit) ordinal $\mu < \omega_1$ such that $K \in \mathcal{PB}^\mu(G)$.

2.4 Lemma: *Let K be a PB^∞-subgroup and $A = \bigcup_{\alpha<\lambda} A_\alpha$ a smooth increasing union of subgroups of a torsion-free group G such that $A_\alpha + K$ is a PB^∞-subgroup of G for each $\alpha < \lambda$. If $\operatorname{cof} \lambda \neq \omega$ and $\mathrm{d}(A_{\alpha+1} + L) \leq \mathrm{d}(L)$ for each $\alpha + 1 < \lambda$ and each $L \in \mathcal{M}(K)$, then $A + L \in PB^\infty(G)$ and $\mathrm{d}(A + L) \leq \mathrm{d}(L)$ for each $L \in \mathcal{M}(K)$.*

Proof: Let $K \in \mathcal{PB}^\mu(G)$. We use transfinite induction on μ. Suppose $\mu = 0$ so that $\mathrm{d}(K) = 0$. Then $A_{\alpha+1} + K$ is G-prebalanced for each $\alpha + 1 < \lambda$ and Lemma 1.2 applies. Suppose $\mu > 0$ and assume the assertion holds for all PB^∞-subgroups K with $\mathrm{d}(K) < \mu$. In view of the definition of the PB^∞-subgroups, we may assume, without loss of generality, that μ is a non-limit ordinal, say, $\mu = \nu + 1$ for some $\nu \geq 0$. Then $K = \bigcup_{n<\omega} K_n$ where $K_n \in \mathcal{PB}^\nu(G)$ and $\mathcal{M}(K) = \bigcup_{n<\omega} \mathcal{M}(K_n) \cup \{K\}$. Consequently, for each $n < \omega$ and for each $L \in \mathcal{M}(K_n) \subset \mathcal{M}(K)$, we have $A_{\alpha+1} + L \in \mathcal{PB}^\infty(G)$ with $\mathrm{d}(A_{\alpha+1} + L) \leq \mathrm{d}(L)$ and this, by induction hypothesis, yields that $A + L \in \mathcal{PB}^\infty(G)$ with $\mathrm{d}(A + L) \leq \mathrm{d}(L)$. In particular, $\mathrm{d}(A + K_n) \leq \mathrm{d}(K_n) \leq \nu$ so that $A + K = \bigcup_{n<\omega}(A + K_n) \in \mathcal{PB}^{\nu+1}(G)$. We are done since $\mathcal{M}(K) = \bigcup_{n<\omega} \mathcal{M}(K_n) \cup \{K\}$. ∎

In the next definition, the concept of compatibility, as defined in [AH] and [DHR], is modified slightly to handle prebalancedness (see also [B]). We shall, however, use the same notation $\|$.

2.5 Definition: If A, B are two subgroups of a torsion-free group G, then $A\|B$ will mean that $A+B$ is pure in G, and for each $a \in A$ and for each $b \in B$, there is $0 < m < \omega$ and $\{c_0, \ldots, c_n\} \subseteq A \cap B$ such that

$$\mathrm{t}(a+b) \leq \bigcup_{i=0}^{n} \mathrm{t}(ma + c_i).$$

If B is a PB^∞-subgroup of G, then we shall simply write $A\|\mathcal{M}(B)$ instead of $A\|K$ for each $K \in \mathcal{M}(B)$. Moreover, for an arbitrary collection $\mathcal{K}$ of PB^∞-subgroups of G, the symbol $A\|\mathcal{M}(\mathcal{K})$ means $A\|K$ for each $K \in \mathcal{M}(\mathcal{K})$.

The following properties have been presented in [B].

2.6 LEMMA: *Let A, B be subgroups of a torsion-free group G. Then*
- (i) *if $A\|B$ then $B\|A$;*
- (ii) *if $A \leq B$ then $A\|B$;*
- (iii) *if $A = \bigcup_{\alpha<\lambda} A_\alpha$, $B = \bigcup_{\alpha<\lambda} B_\alpha$ are smooth unions and $A_\alpha\|B_\alpha$ for each $\alpha < \lambda$, then $A\|B$;*
- (iv) *in particular, if $B = \bigcup_{\alpha<\lambda} B_\alpha$ and $A\|B_\alpha$ for each $\alpha < \lambda$, then $A\|B$.*

2.7 LEMMA: *If B is G-prebalanced and $A\|B$, then $A \cap B$ is A-prebalanced.*

Proof: See [B; L.2.8], or [DHR; L.7.2] in the "balanced" case. ∎

2.8 LEMMA: *If $B \in \mathcal{PB}^\nu(G)$ for some ordinal ν and A is a subgroup of G such that $A\|B$, then $A \cap B \in \mathcal{PB}^\nu(A)$. Moreover, $\mathrm{d}(A \cap B) \leq \mathrm{d}(B)$.*

Proof: We apply transfinite induction on ν. The case when $\nu = 0$ has been treated in Lemma 2.7. Suppose $\nu > 0$ and assume the lemma holds for all $\mu < \nu$. The case when ν is a limit ordinal presents no difficulty. So let $\nu = \mu + 1$ and we can also assume that $\nu = \mathrm{d}(B)$. Then $B = \bigcup_{n<\omega} B_n$ with $B_n \in \mathcal{PB}^\mu(G)$. Since $A\|\mathcal{M}(B_n)$ for each $n < \omega$, the induction hypothesis yields that $A \cap B_n \in \mathcal{PB}^\mu(A)$ and $\mathrm{d}(A \cap B_n) \leq \mathrm{d}(B_n) \leq \mu$. Then $A \cap B = \bigcup_{n<\omega}(A \cap B_n) \in \mathcal{PB}^{\mu+1}(A)$ and $\mathrm{d}(A \cap B) \leq \mu + 1 = \mathrm{d}(B)$. ∎

2.9 LEMMA: *If $A\|B$ and $A+B\|C$, then $B\|A+C$.*

Proof: See [B; L.2.8], or [DHR; L.7.2] in the "balanced" case. ∎

3. Prespecial subsets and PB^∞-subgroups

Let K be a pure subgroup of a completely decomposable group $C = \bigoplus_{i\in M} X_i$ with X_i rank one and torsion-free. For each $J \subseteq M$ we shall use the following notations:

$$C(J) = \bigoplus_{i\in J} X_i, \;\; K(J) = C(J) \cap K$$

and $\varphi_i\colon C \to X_i$ will denote the canonical projection.

3.1 Definition: Let K be a pure subgroup of a completely decomposable group $C = \bigoplus_{i\in M} X_i$. A subset $J \subseteq M$ is said to be K-prespecial if $C(J)\| K$. If K is a PB^∞-subgroup of C, then we say that J is $\mathcal{M}(K)$-prespecial if $C(J)\|\mathcal{M}(K)$. Moreover, for an arbitrary collection $\mathcal{K}$ of PB^∞-subgroups of C, we say that J is $\mathcal{M}(\mathcal{K})$-prespecial if $C(J)\|\mathcal{M}(\mathcal{K})$.

3.2 Lemma: *Let $\mathcal{K}$ be a countable collection of PB^∞-subgroups of a completely decomposable group $C = \bigoplus_{i\in M} X_i$. If $X \subseteq C$ is any infinite subset, then there is a subset $J \subseteq M$ such that $|J| = |X|$, $X \subseteq C(J)$ and J is $\mathcal{M}(\mathcal{K})$-prespecial.*

Proof: See [B; L.3.2]. ∎

3.3 Definition: Let $\mathcal{K}$ be any collection of PB^∞-subgroups of a torsion-free group G. We say, that a subgroup H of G is an $\mathcal{M}(\mathcal{K})$-subgroup if

(1) $H \| \mathcal{M}(\mathcal{K})$;

(2) $H+K$ is a PB^∞-subgroup of G and $\mathrm{d}(H+K) \leq \mathrm{d}(K)$ for each $K \in \mathcal{M}(\mathcal{K})$.

Moreover, if instead of (2),

(2+) $H + K$ is a PB^∞-subgroup of G and $\mathrm{d}(H + K) \leq \mathrm{d}(K) + 1$ for each $K \in \mathcal{M}(\mathcal{K})$

holds, then we say that H is an $\mathcal{M}^+(\mathcal{K})$-subgroup of G.

3.4 Lemma: *Let $\mathcal{K}$ be any collection of PB^∞-subgroups and $A = \bigcup_{\alpha<\lambda} A_\alpha$ be a smooth increasing union of subgroups of a torsion-free group G. If $A_{\alpha+1}$ is an $\mathcal{M}(\mathcal{K})$-subgroup of G for each $\alpha + 1 < \lambda$, then A is an $\mathcal{M}(\mathcal{K})$-subgroup of G whenever* cof $\lambda \neq \omega$ *and it is an $\mathcal{M}^+(\mathcal{K})$-subgroup of G otherwise.*

Proof: The condition (1) follows immediately from Lemma 2.6. An appeal to Lemma 2.4 proves (2) in the case cof $\lambda \neq \omega$, while the case cof $\lambda = \omega$ is easy to verify. ∎

3.5 Definition: Let $\mathcal{K}$ be any collection of PB^∞-subgroups of a completely decomposable group $C = \bigoplus_{i\in M} X_i$. We say that $J \subseteq M$ is an $\mathcal{M}(\mathcal{K})$-set ($\mathcal{M}^+(\mathcal{K})$-set) if $C(J)$ is an $\mathcal{M}(\mathcal{K})$-subgroup ($\mathcal{M}^+(\mathcal{K})$-subgroup) of C.

3.6 LEMMA (GCH): *Let K be prebalanced subgroup of a completely decomposable group $C = \bigoplus_{i\in M} X_i$. If $X \subseteq C$ is any subset of uncountable cardinality not cofinal to ω, then there is $J \subseteq M$ such that $|J| = |X|, X \subseteq C(J)$ and $C(J) + K$ is C-prebalanced.*

Proof: Set $J_0 = \{i \in M \mid \varphi_i(X) \neq 0\}$. If J_α with $|J_\alpha| = |X|$ has been constructed for some $\alpha < \omega_1$, then we have $|(C(J_\alpha) + K)/K| \leq |C(J_\alpha)| = |X|$ and consequently, by Lemma 1.3, there is a C/K-prebalanced subgroup L_α/K containing $(C(J_\alpha) + K)/K$ and of cardinality not exceeding $|X|$. Clearly we can write $L_\alpha = \widetilde{L_\alpha} + K$, where $|\widetilde{L_\alpha}| \leq |L_\alpha/K| = |X|$ and we set

$$J_{\alpha+1} = J_\alpha \cup \{i \in M \mid \varphi_i(\widetilde{L_\alpha}) \neq 0\}.$$

If $\alpha < \omega_1$ is limit and J_β has been defined for all $\beta < \alpha$ with $J_\gamma \subseteq J_\beta$ for $\gamma \leq \beta < \alpha$, then we set $J_\alpha = \bigcup_{\beta<\alpha} J_\beta$.

Obviously, the union $J = \bigcup_{\alpha<\omega_1} J_\alpha$ is of cardinality $|X|$ and $C(J) + K = \bigcup_{\alpha<\omega_1}(C(J_\alpha) + K) \leq \bigcup_{\alpha<\omega_1} L_\alpha \leq \bigcup_{\alpha<\omega_1}(C(J_{\alpha+1}) + K) = C(J) + K$ is C-prebalanced by Lemma 1.2. ∎

3.7 LEMMA (GCH): *Let $C = \bigoplus_{i\in M} X_i$ be a completely decomposable group and $K \in \mathcal{PB}^\nu(C)$ for some ordinal ν. If $X \subseteq C$ is any subset of uncountable cardinality not cofinal to ω, then there is a $J \subseteq M$ such that $|J| = |X|$, $X \subseteq C(J)$ and $C(J) + K$ is a PB^∞-subgroup of C with $\mathrm{d}(C(J) + K) \leq \mathrm{d}(K)$.*

Proof: We apply induction on ν, the result being true when $\nu = 0$ by Lemma 3.6. We need only consider the case when ν is not a limit ordinal, say, $\nu = \mu + 1$. We may also take $\mathrm{d}(K) = \mu + 1$. Then $K = \bigcup_{n<\omega} K_n$ with $\mathrm{d}(K_n) \leq \mu$. Now, for each $n < \omega$, we can construct a subset J_n of M such that $|J_n| = |X|$, $X \subseteq C(J_n)$, $J_n \subseteq J_{n+1}$, and $C(J_n) + K_n$ is a PB^∞-subgroup of C with $\mathrm{d}(C(J_n) + K_n) \leq \mathrm{d}(K_n)$. If we set $J = \bigcup_{n<\omega} J_n$, we see that $C(J) + K = \bigcup_{n<\omega}(C(J_n) + K_n) \in \mathcal{PB}^{\mu+1}(C)$. ∎

3.8 LEMMA (GCH): *Let $\mathcal{K}$ be a set of PB^∞-subgroups of a completely decomposable group $C = \bigoplus_{i\in M} X_i$ with $|\mathcal{K}| \leq \aleph_1$. If $X \subseteq C$ is any subset of uncountable* *cardinality not cofinal to* ω, *then there is* $J \subseteq M$ *such that* $|J| = |X|, X \subseteq C(J)$, *and* $\mathrm{d}(C(J) + K) \le \mathrm{d}(K)$ *for each* $K \in \mathcal{M}(\mathcal{K})$.

Proof: Since $|\mathcal{M}(\mathcal{K})| \le \aleph_1$ and since there are $\aleph_1$-many pair-wise disjoint cofinal subsets of ω_1, we may assume that $\{K_\alpha \mid \alpha < \omega_1\}$ is an arrangement of elements of $\mathcal{M}(\mathcal{K})$ in which each distinct member of $\mathcal{M}(\mathcal{K})$ occurs uncountably many times indexed by a cofinal subset of ω_1. Set $J_0 = \{i \in M \mid \varphi_i(X) \neq 0\}$. If $\alpha < \omega_1$ is limit and J_β with cardinality $|X|$ has been defined for all $\beta < \alpha$ in such a way that $J_\gamma \subseteq J_\beta$ whenever $\gamma \le \beta < \alpha$, then we set $J_\alpha = \bigcup_{\beta<\alpha} J_\beta$. If $\alpha = \beta + 1$ and J_β is defined with $|J_\beta| = |X|$, then by Lemma 3.7 there is $J_{\beta+1} \subseteq M$ such that $|J_{\beta+1}| = |X|, C(J_\beta) \le C(J_{\beta+1})$ and $\mathrm{d}(C(J_{\beta+1}) + K_\beta) \le \mathrm{d}(K_\beta)$. Now if we set $J = \cup_{\alpha<\omega_1} J_\alpha$, then obviously $|J| = |X|$ and $X \subseteq C(J)$. If $K \in \mathcal{M}(\mathcal{K})$ is arbitrary, then there is a cofinal subset $\{i_\alpha \mid \alpha < \omega_1\} \subseteq \omega_1$ such that $K_{i_\alpha} = K$ for each $\alpha < \omega_1$. Now the equality $C(J) + K = \bigcup_{\alpha<\omega_1}(C(J_{i_\alpha+1}) + K)$ shows that $\mathrm{d}(C(J) + K) \le \mathrm{d}(K)$ owing to Lemma 2.4. ∎

4. The key results

4.1 Lemma (GCH): *Let* $\mathcal{K}$ *be a countable collection of PB$^\infty$-subgroups of a completely decomposable group* $C = \bigoplus_{i\in M} X_i$. *If* $X \subseteq C$ *is any subset of uncountable cardinality not cofinal to* ω, *then there is* $J \subseteq M$ *such that*

(i) $X \subseteq C(J)$ *and* $|J| = |X|$;

(ii) J *is an* $\mathcal{M}(\mathcal{K})$*-set.*

Proof: Set $J_0^h = J_0^p = \{i \in M \mid \varphi_i(X) \neq 0\}$ and assume that the sets J_β^h, J_β^p have been defined for each $\beta < \alpha < \omega_1$ in such a way that they are all of the same cardinality $|X|$ and $J_\beta^p \subseteq J_\beta^h$. For $\alpha = \beta + 1$ there is, by Lemma 3.2, $J_\alpha^p \subseteq M$ of cardinality $|X|$ containing J_β^h and such that J_α^p is $\mathcal{M}(\mathcal{K})$-prespecial. Further, Lemma 3.8 yields the existence of $J_\alpha^h \subseteq M$ of cardinality $|X|$ containing J_α^p and such that $\mathrm{d}(C(J_\alpha^h) + K) \le \mathrm{d}(K)$ for each $K \in \mathcal{M}(\mathcal{K})$. For a limit ordinal $\alpha < \omega_1$, we simply set $J_\alpha^h = J_\alpha^p = \bigcup_{\beta<\alpha} J_\beta^h = \bigcup_{\beta<\alpha} J_\beta^p$. Finally we set $J = \bigcup_{\alpha<\omega_1} J_\alpha^p = \bigcup_{\alpha<\omega_1} J_\alpha^h$. Then for each $K \in \mathcal{M}(\mathcal{K})$ we have $\mathrm{d}(C(J) + K) = \mathrm{d}(\bigcup_{\alpha<\omega_1}(C(J_\alpha^h) + K)) \le \mathrm{d}(K)$ by Lemma 2.4 and so (ii) is true. The proof is complete, (i) being obvious. ∎

4.2 Lemma (GCH): *Let* $\mathcal{K}$ *be a countable collection of PB$^\infty$-subgroups of a completely decomposable group* $C = \bigoplus_{i\in M} X_i$ *of the cardinality* $|M|$ *which is* *not a successor of a cardinal of cofinality ω. Then M is a smooth increasing union $M = \bigcup_{\alpha<\lambda} J_\alpha$, $\lambda = \operatorname{cof} |M|$, such that, for each $\alpha < \lambda$, $|J_\alpha| < |M|$, J_α is an $\mathcal{M}^+(\mathcal{K})$-set or an $\mathcal{M}(\mathcal{K})$-set according as $\operatorname{cof} \alpha = \omega$ or not and $J_{\alpha+1}$ is a $(C(J_\alpha) + \mathcal{M}(\mathcal{K}))$-set.*

Proof: For $|M|$ limit we can write $|M| = \sum_{\alpha<\lambda} \kappa_\alpha$, where κ_0 and $\kappa_{\alpha+1}$ are uncountable regular cardinals for each $\alpha < \lambda$. If $|M| = \lambda = \kappa^+$ with $\operatorname{cof} \kappa \neq \omega$, we set $\kappa_\alpha = \kappa$ for each $\alpha < \lambda$ in this case. So, in both cases we can write $M = \bigcup_{\alpha<\lambda} I_\alpha$, where $|I_\alpha| = \kappa_\alpha$ for each $\alpha < \lambda$. According to Lemma 4.1 we select $J_0 \subseteq M$ containing I_0 of the cardinality κ_0 and we shall continue by the transfinite induction.

For α limit we simply set $J_\alpha = \bigcup_{\beta<\alpha} J_\beta$ and Lemma 3.4 shows that J_α is an $\mathcal{M}(\mathcal{K})$-set whenever $\operatorname{cof} \alpha \neq \omega$. The case of $\operatorname{cof} \alpha = \omega$ is obvious. If $\alpha = \gamma + 1$ and J_γ is defined as an $\mathcal{M}(\mathcal{K})$-set containing I_γ and $|J_\gamma| = \kappa_\gamma$, then we select J_α to be a subset of M corresponding to $\mathcal{K} \cup (C(J_\gamma) + \mathcal{M}(\mathcal{K}))$ and $X = C(J_\gamma \cup I_\alpha)$ by Lemma 4.1. Then J_α is obviously an $\mathcal{M}(\mathcal{K})$-set as well as a $(C(J_\gamma) + \mathcal{M}(\mathcal{K}))$-set and the proof is finished. ∎

As was mentioned in [FMa], the main difficulty in our inductive argument lies at the cardinality which is the successor of a singular cardinal of cofinality ω. As in the cited paper we shall need the following Jensen's Principle holding in the constructible universe.

$\square_\kappa$: If κ is a singular cardinal, then there exists a family of sets D_α, indexed by limit ordinals $\alpha < \kappa^+$, such that

(i) D_α is closed and unbounded in α;

(ii) the order type of D_α is $< \kappa$;

(iii) $D_\beta = D_\alpha \cap \beta$ whenever β is a limit point of D_α (coherence property).

If κ is of cofinality ω, then $\kappa = \sum_{m<\omega} \kappa_m$, where κ_m are regular cardinals (cf. [J]). Now we are ready to present the crucial step in deriving our result.

4.3 Lemma (V = L): *Let $\mathcal{K}$ be a countable collection of PB^∞-subgroups of a completely decomposable group $C = \bigoplus_{i\in M} X_i$ of cardinality κ^+, where κ is singular with $\operatorname{cof} \kappa = \omega$. Then M is a smooth increasing union $M = \bigcup_{\alpha<\kappa^+} J_\alpha$, such that $|J_\alpha| = \kappa$, J_α is an $\mathcal{M}^+(\mathcal{K})$-set and $J_{\alpha+1}$ is a $(C(J_\alpha) + \mathcal{M}^+(\mathcal{K}))$-set for each $\alpha < \kappa^+$.*

Proof: We shall essentially follow the ideas of the proof of the key lemma 3.1 in [FMa]. Let $M = \{m_\alpha \mid \alpha < \kappa^+\}$ be a well-ordering of elements of M.

We are going to show that M can be written as a smooth increasing union $M = \bigcup_{\alpha<\kappa^+} J_\alpha$ in such a way that

(1) $J_\alpha = \bigcup_{m<\omega} J_\alpha^m$ such that $J_\alpha^m \subseteq J_\alpha^{m+1}$ and $|J_\alpha^m| = \kappa_m$ for all $\alpha < \kappa^+$ and $m < \omega$;

(2) for every $\beta < \alpha < \kappa^+$, $J_\beta^m \subseteq J_\alpha^m$ if $m < \omega$ is large enough;

(3) if $\alpha = \beta + 1$, then $m_\beta \in J_\alpha$ and $J_\beta^m \subseteq J_\alpha^m$ for each $m < \omega$;

(4) if $\alpha = \beta + 1$ and β is a limit ordinal such that the order type of D_β is $\geq \kappa_m$, then $J_\beta^m = J_\alpha^m$;

(5) if β is a limit point of D_α, then $J_{\beta+1}^m \subseteq J_\alpha^m$ for each $m < \omega$;

(6) if β is a limit point of D_α and if the order type of D_β is $\geq \kappa_m$, then $J_\beta^m = J_\alpha^m$;

(7) J_α^m is an $\mathcal{M}(\mathcal{K})$-set for each $m < \omega$ and for each $\alpha < \kappa^+$ such that either (i) cof $\alpha \neq \omega$ or (ii) cof $\alpha = \omega$ and the set of limit points in D_α is bounded in α; otherwise J_α^m is an $\mathcal{M}^+(\mathcal{K})$-set.

(8) $J_{\alpha+1}^m$ is a $(C(J_\alpha^m) + \mathcal{M}(\mathcal{K}))$-set for each $m < \omega$ and each $\alpha < \kappa^+$ for which either (i) cof $\alpha \neq \omega$ or (ii) cof $\alpha = \omega$ and the set of limit points in D_α is bounded in α; otherwise it is an $(C(J_\alpha^m) + \mathcal{M}^+(\mathcal{K}))$-set.

We apply induction on α.

CASE 1: Let $\alpha = 0$. For each $m < \omega$, set $M_m = \{m_\alpha \mid \alpha < \kappa_m\}$. Now we take J_0^0 according to Lemma 4.1 corresponding to $X = C(M_0)$. Assuming J_0^m has been defined for some $m < \omega$, we select J_0^{m+1} according to Lemma 4.1 for $X = C(J_0^m \cup M_{m+1})$. Setting $J_0 = \bigcup_{m<\omega} J_0^m$, the assertions (2) – (6), (8) are vacuous in this case, while the other ones are trivial.

CASE 2: α is a limit ordinal and the set of limit points in D_α is bounded in α.

In this case, Jensen's principle (i) implies that the order type of D_α is $\delta + \omega$, where either $\delta = 0$ or δ is a limit ordinal. In particular, cof $\alpha = \omega$. Set η to be the largest limit ordinal in D_α (it is the δ-th term) if it exists and set $\eta = 0$ otherwise.

Put $\beta_0 = \eta + 1$ and choose a sequence $\beta_0 < \beta_1 < \cdots$ of successor ordinals tending to α. Let m_0 be the smallest t with $\delta < \kappa_t$. If m_{i-1} has been defined, then since β_i satisfies (2) by induction, we can take m_i to be the smallest integer with $m_i > m_{i-1}$ and

$$J_{\beta_j}^m \subseteq J_{\beta_i}^m \quad \text{for all } j < i \text{ and } m \geq m_i$$

(such m_i exists since, by induction, β_i satisfies (2)). This way we obtain the sequence $\{m_i \mid i < \omega\}$.

Now we set

$$J_\alpha^m = \begin{cases} J_{\beta_0}^m & \text{if } m < m_0, \\ J_{\beta_i}^m & \text{if } m_i \leq m < m_{i+1} \text{ for some } i < \omega, \end{cases}$$

and then define $J_\alpha = \bigcup_{m<\omega} J_\alpha^m$.

We shall verify the smoothness. Clearly, $J_\alpha = \bigcup_{m<\omega} J_\alpha^m \subseteq \bigcup_{m<\omega} \bigcup_{i<\omega} J_{\beta_i}^m \subseteq \bigcup_{\beta<\alpha} \bigcup_{m<\omega} J_\beta^m = \bigcup_{\beta<\alpha} J_\beta$. Conversely, if $x \in J_\beta$ for $\beta < \alpha$, we have $x \in J_\beta^m$ for suitable $m < \omega$. There is a $j < \omega$ with $\beta < \beta_j$ and so $x \in J_{\beta_j}^m$. Now there exists i such that $m_i \leq m < m_{i+1}$ and, without loss of generality, we could assume that $j < i$. Now $J_{\beta_j}^m \subseteq J_{\beta_i}^m = J_\alpha^m \subseteq J_\alpha$.

Turning to (1), for a given m, let i be such that $m_i \leq m < m_{i+1}$. For $m + 1 < m_{i+1}$ we have $J_\alpha^m = J_{\beta_i}^m \subseteq J_{\beta_i}^{m+1} = J_\alpha^{m+1}$, while for $m + 1 = m_{i+1}$ we get $J_\alpha^m = J_{\beta_i}^m \subseteq J_{\beta_i}^{m+1} = J_{\beta_i}^{m_{i+1}} = J_\alpha^{m+1}$.

As for (2), let $\beta < \alpha$ be arbitrary. Then $\beta < \beta_j$ for some $j < \omega$, $J_\beta^m \subseteq J_{\beta_j}^m$ for m large enough and it suffices to show that $J_{\beta_j}^m \subseteq J_\alpha^m$ for $m \geq m_j$. But there is $j \leq i < \omega$ with $m_i \leq m < m_{i+1}$ and so $J_{\beta_j}^m \subseteq J_{\beta_i}^m = J_\alpha^m$.

We next verify (5) since (3) and (4) are vacuous in this case. If β is a limit point of D_α, then $\beta \leq \eta$. If $\beta = \eta$, then for $m < m_0$ we have $J_{\beta+1}^m = J_{\beta_0}^m = J_\alpha^m$ by the choice of β_0. Further, $m_i \leq m < m_{i+1}$ yields $J_{\beta+1}^m = J_{\beta_0}^m \subseteq J_{\beta_i}^m = J_\alpha^m$ and we are done. Suppose $\beta < \eta$. In this case β is a limit point of $D_\eta = D_\alpha \cap \eta$, hence, by the induction hypothesis, $J_{\beta+1}^m \subseteq J_\eta^m \subseteq J_{\eta+1}^m = J_{\beta_0}^m \subseteq J_\alpha^m$ for each $m < \omega$ by (3) and the above part.

In order to check (6), recall that m_0 was chosen as a minimal t with $\delta < \kappa_t$ and η is the δ-th term in D_α. Hence δ is the order type of D_η. If β is a limit point of D_α, then the order type of $D_\beta = D_\alpha \cap \beta$ is at most δ and so if it is $\geq \kappa_m$, then necessarily $m < m_0$. If $\beta < \eta$, then β is a limit point of D_η and so $J_\beta^m = J_\eta^m$ by the induction hypothesis. For $\beta = \eta$ the inequality $m < m_0$ and (4) yields $J_\beta^m = J_{\beta+1}^m = J_{\beta_0}^m = J_\alpha^m$.

Finally, (7) and the first assertion are trivial by the choice of J_α^m and the induction hypothesis, while (8) and the second assertion are vacuous in this case.

CASE 3: α is a limit ordinal and the set L_α of limit points in D_α is unbounded in α. We now define

$$J_\alpha^m = \bigcup_{\beta \in L_\alpha} J_{\beta+1}^m \quad \text{and} \quad J_\alpha = \bigcup_{m<\omega} J_\alpha^m.$$

Observe that if $\gamma < \beta$ in L_α, then γ is a limit point of D_β and so, by (5) and (3), $J^m_{\gamma+1} \subseteq J^m_\beta \subseteq J^m_{\beta+1}$ for each $m < \omega$ and that the smoothness immediately follows from the definition of J_α.

As for (1) we clearly have $J^m_\alpha = \bigcup_{\beta\in L_\alpha} J^m_{\beta+1} \subseteq \bigcup_{\beta\in L_\alpha} J^{m+1}_{\beta+1} = J^{m+1}_\alpha$ for each $m < \omega$ and we are going to verify the cardinality property. Again, let t be the smallest integer such that the order type of D_α is $< \kappa_t$. For $m \geq t$ we see that $J^m_\alpha = \bigcup_{\beta\in L_\alpha} J^m_{\beta+1}$ is a union of at most κ_t sets of cardinalities κ_m, hence $|J^m_\alpha| = \kappa_m$. If $m < t$, then the order type of D_α is $> \kappa_m$. Let β be the κ_m-th member of D_α and let $\beta < \gamma \in L_\alpha$. Then $D_\beta = D_\gamma \cap \beta$ yields $\beta \in L_\gamma$ and so (4) and (6) give $J^m_{\beta+1} = J^m_\beta = J^m_\gamma = J^m_{\gamma+1}$ from which it clearly follows that $|J^m_\alpha| = \kappa_m$. Moreover, $J^m_\alpha = J^m_\beta$ proves (6).

To verify (2), take $\beta < \alpha$ arbitrarily. There is a $\gamma \in L_\alpha$ such that $\beta < \gamma < \alpha$, and so $J^m_\beta \subseteq J^m_\gamma$ for m large enough by the induction hypothesis. However, $J^m_\gamma \subseteq J^m_{\gamma+1} \subseteq J^m_\alpha$ by (3) and the definition of J^m_α.

We are going to prove only the first assertion of (7) because (3), (4), (8) and the second assertion of (7) are vacuous in this case, while (5) is an immediate consequence of the definition of J^m_α. In view of the hypothesis in Case 3, we need only consider the case when $\mathrm{cof}(\alpha) \neq \omega$. The condition (1) from Definition 3.3 follows immediately from induction and Lemma 2.6. In view of the definition of J^m_α and the inductive hypothesis, an appeal to Lemma 3.4 shows that J^m_α is an $\mathcal{M}(\mathcal{K})$-set.

Case 4: $\alpha = \beta + 1$. For β non-limit we set $t = 0$, and if β is a limit ordinal, let t be the smallest integer such that the order type of D_β is $< \kappa_t$ (which exists by $\square_\kappa$ (ii)).

If $m < t$, we set $J^m_\alpha = J^m_\beta$. Suppose $m \geq t$. We define J^m_α to be the set J given by Lemma 4.1 where we take for X the set $X = C(J^{m-1}_\alpha \cup J^m_\beta \cup \{m_\beta\})$ (assuming obviously $J^{-1}_\alpha = \emptyset$) and for the family $\mathcal{K}$ we make the following choices: If $\mathrm{cof}\,\beta \neq \omega$ or if $\mathrm{cof}\,\beta = \omega$ and the set L_β of limit points in D_α is bounded in β, then take $\mathcal{K} \cup (C(J^m_\beta) + \mathcal{M}(\mathcal{K}))$ in place of $\mathcal{K}$ in Lemma 4.1. If β is the limit of a sequence $\{\beta_k \mid k < \omega\} \subseteq L_\beta$, then take $\mathcal{K} \cup \bigcup_{k<\omega}(C(J^m_{\beta_k+1}) + \mathcal{M}(\mathcal{K}))$ for the family $\mathcal{K}$ in Lemma 4.1. Finally we set $J_\alpha = \bigcup_{m<\omega} J^m_\alpha$. We need to verify all the properties stated.

If β is a limit ordinal and the order type of D_β is $> \kappa_m$, then $m < t$, and $J^m_\beta = J^m_\alpha$ by the construction and so (4) holds. The conditions (5), (6) are vacuously true and the remaining conditions are easy to verify. ∎

4.4 PROPOSITION (V= L): *Let $\mathcal{K}$ be a countable collection of PB^∞-subgroups of a completely decomposable group $C = \bigoplus_{i\in M} X_i$. Then M is a smooth increasing union $M = \bigcup_{\alpha<\lambda} J_\alpha$, $\lambda = \mathrm{cof}\ |M|$, such that $|J_\alpha| < |M|$, J_α is an $\mathcal{M}^+(\mathcal{K})$-set and $J_{\alpha+1}$ is an $(C(J_\alpha) + \mathcal{M}^+(\mathcal{K}))$-set for each $\alpha < \lambda$.*

Proof: An immediate consequence of two preceding lemmas. ∎

5. Main results

5.1 THEOREM (V = L): *Any PB^∞-subgroup of a completely decomposable group is a B_1-group.*

Proof: Proving indirectly, let us suppose that $C = \bigoplus_{i\in M} X_i$ is a completely decomposable group of the smallest possible cardinality $\kappa = |M|$ containing a PB^∞-subgroup K which is not a B_1-group. By [BS], $\kappa > \aleph_0$.

By Proposition 4.4 we have $M = \bigcup_{\alpha<\lambda} J_\alpha$, $\lambda = \mathrm{cof}\ \kappa$ and $|J_\alpha| < \kappa$. Since $C(J_\alpha)\|\mathcal{M}(K)$, $K(J_\alpha) = C(J_\alpha)\cap K$ is a PB^∞-subgroup of $C(J_\alpha)$ by Lemma 2.8 and consequently it is a B_1-group by the choice of K.

The desired contradiction can be obtained from 1.5 provided we show that $K(J_\alpha)$ is TEP in $K(J_{\alpha+1})$ for each $\alpha < \lambda$. Now $C(J_\alpha)\|K$ and so Lemma 2.7 yields that $K(J_\alpha)$ is K-prebalanced and consequently it is $K(J_{\alpha+1})$-prebalanced. Looking at Lemma 1.4(ii) we infer that it suffices to show that $K(J_{\alpha+1})/K(J_\alpha)$ is a B_1-group.

The direct summand $C(J_\alpha)$ is trivially TEP in $C(J_{\alpha+1}) \cap (C(J_\alpha) + K)$ and so, in view of Lemma 1.4(i) and the fact that $K(J_{\alpha+1})/K(J_\alpha) = (C(J_{\alpha+1})\cap K)/(C(J_\alpha)\cap K) \cong ((C(J_{\alpha+1})\cap K) + C(J_\alpha))/C(J_\alpha)$, it suffices to show that $(C(J_{\alpha+1})\cap K) + C(J_\alpha) = C(J_{\alpha+1})\cap(C(J_\alpha)+K)$ is a B_1-group. But $C(J_\alpha) + K$ is a PB^∞-subgroup of C, $J_{\alpha+1}$ is a $(C(J_\alpha) + \mathcal{M}^+(\mathcal{K}))$-set and so $C(J_{\alpha+1})\cap(C(J_\alpha)+K) \in \mathcal{PB}^\infty(C(J_{\alpha+1}))$ by Lemma 2.8. Hence it is a B_1-group by the choice of K. This means K is a B_1-group — a contradiction. ∎

5.2 LEMMA: *Let*

$$\begin{array}{ccccccccc} E\colon 0 & \longrightarrow & K & \longrightarrow & D & \longrightarrow & A & \longrightarrow & 0 \\ & & \| & & \downarrow & & \downarrow{\scriptstyle\iota} & & \\ F\colon 0 & \longrightarrow & K & \longrightarrow & C & \longrightarrow & B & \longrightarrow & 0 \end{array}$$

be a commutative diagram with exact rows, F prebalanced and ι the inclusion map. Then $A \in \mathcal{PB}^\infty(B)$ if and only if $D \in \mathcal{PB}^\infty(C)$. Moreover, we have $\mathrm{d}(A) = \mathrm{d}(D)$ in this case.

Proof: Clearly, from the prebalancedness of F we infer that D is C-prebalanced if and only if A is B-prebalanced and so the assertion holds for $\mathrm{d}(A) = \mathrm{d}(D) = 0$. So, if $A \in \mathcal{PB}^\mu(B)$ ($D \in \mathcal{PB}^\mu(C)$), we can use transfinite induction on μ. In view of the definition of the PB^∞-subgroups, we may assume, without loss of generality, that μ is a non-limit ordinal, say, $\mu = \nu + 1$ for some $\nu \geq 0$. Then $A = \bigcup_{n<\omega} A_n$ ($D = \bigcup_{n<\omega} D_n$) with $\mathrm{d}(A_n) \leq \nu$ ($\mathrm{d}(D_n) \leq \nu$) for each $n < \omega$. Thus $\mathrm{d}(D_n) = \mathrm{d}(A_n)$ for each $n < \omega$ by the induction hypothesis and so $\mathrm{d}(D) = \mathrm{d}(A) = \nu + 1 = \mu$, as desired. ∎

Our next result is a direct generalization of [BF; Th.4.5] and it is closely related to [B; Th.8.3].

5.3 THEOREM (V = L): *Any PB^∞-subgroup of a B_1-group is a B_1-group.*

Proof: Let A be a PB^∞-subgroup of a B_1-group B. Considering the commutative diagram from the preceding lemma where F is a balanced-projective resolution of B, we see that, in view of Lemma 1.4, both E and F are prebalanced and TEP. Since D is a PB^∞-subgroup of C by Lemma 5.2, D is a B_1-group by Theorem 5.1 and Lemma 1.4 finishes the proof. ∎

The following result generalizes [DHR; Cor.5.9] and [B; Th.8.6].

5.4 THEOREM (CH): *Any PB^∞-subgroup of a B_1-group of cardinality not exceeding $\aleph_\omega$ is a B_1-group.*

Proof: Under this cardinality restriction, (CH) is enough in the proof of Lemma 4.2 which is clearly sufficient in the proof of Theorem 5.1. ∎

Finally, if B is a B_1-group of cardinality $\leq \aleph_1$, then Theorem 5.3 holds without any additional set-theoretical hypothesis. Since a PB^∞-subgroup is always preseparative, we establish this result as a consequence of the following more general theorem which is valid under ZFC.

5.5 THEOREM: *Suppose there is a smooth preseparative chain $A = A_0 \leq \cdots \leq A_\alpha \leq \cdots B = \bigcup_{\alpha<\lambda} A_\alpha$ with $A_{\alpha+1}/A_\alpha$ countable for each $\alpha < \lambda$. If B is a B_1-group, then so is A.*

Proof: For any torsion group T, suppose for some $\alpha \geq 0$ we have a balanced exact sequence $0 \to T \to H_\alpha \xrightarrow{\varphi_\alpha} A_\alpha \to 0$. Consider the diagram

$$\begin{tikzcd}
 & & K \arrow[r, equal] \arrow[d] & K \arrow[d] & \\
0 \arrow[r] & T \arrow[r] \arrow[d, equal] & H_\alpha \oplus C \arrow[r] \arrow[d] & A_\alpha \oplus C \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & T \arrow[r] & H_{\alpha+1} \arrow[r] & A_{\alpha+1} \arrow[r] & 0 ,
\end{tikzcd}$$

where C is a countable completely decomposable group and K a B_2-group by [BF]. So there is a homomorphism $K \to H_\alpha \oplus C$ which, composed with $\varphi_\alpha \oplus 1_C$, equals the embedding of K into $A_\alpha \oplus C$. Let $H_{\alpha+1} = (H_\alpha \oplus C)/K$. The bottom row is clearly balanced. Thus we get a direct system of balanced exact sequences $0 \to T \to H_\alpha \to A_\alpha \to 0$, where the balancedness at limit ordinals follows from the fact that the A_α are all torsion-free and pure in $A_{\alpha+1}$. For the same reason, the direct limit $0 \to T \to H \to B \to 0$ is balanced exact and hence splits. This means that each A_α and, in particular, A is a B_1-group. ∎

Remark: The proof of the above theorem follows the ideas from the paper of Fuchs and Viljoen [FV].

5.6 THEOREM: *If B is a B_1-group and A is a preseparative subgroup with $|B/A| \leq \aleph_1$, then A is a B_1-group.*

Proof: Let $A = A_0 \leq A_1 \leq \cdots \leq A_\alpha \leq \cdots, \alpha < \omega_1$, be an ascending chain of pure subgroups so that $B = \bigcup_{\alpha<\omega_1} A_\alpha$, and, for each α, $A_{\alpha+1}/A_\alpha$ is countable. Now each A_α is preseparative as a countable extension of a preseparative subgroup and Theorem 5.5 applies. ∎

5.7 COROLLARY: *In a B_1-group B of cardinality at most $\aleph_1$ every preseparative subgroup (in particular, any PB^∞-subgroup) is again a B_1-group.* ∎

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